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Improved bound for the Carathéodory rank of the bases of a matroid ☆

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Abstract

Let M be a matroid on m elements and let r be its rank. We show that any vector in the integer cone of the incidence vectors of bases of M can be written as nonnegative integer combination of at most $m+r-1$ incidence vectors of bases of M.

1. Introduction

Let H be a set of vectors in $\mathbb{Z}^m$. We denote by cone(H), lat(H), and int.cone(H) the cone, the lattice, and the integer cone generated by the vectors in H, respectively (for undefined terms see [9]). We have that *int.cone*$(H) \subseteq$ cone$(H) \cap$ lat(H). The set H is called a *Hilbert base* if equality holds. The *Carathéodory rank* of H is the least integer t such that any vector in int.cone(H) can be written as a nonnegative integer combination of at most t vectors in H.

Cook et al. [2] have proved that any Hilbert base in $\mathbb{Z}^m$ generating a pointed cone has Carathéodory rank at most $2m-1$. Sebő [11] improved this bound to $2m-2$ and conjectured that the actual upper bound is m. This conjectured has been disproved by Bruns et al. [1].

It follows from Edmonds' Matroid Partition Theorem [3] that incidence vectors of the bases of a matroid form a Hilbert base generating a pointed cone. So, Sebő's result implies the upper bound of $2m-2$ for Carathéodory rank of any Hilbert bases arising from the bases of a matroid on m elements.

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In this paper, we prove the following theorem.

Theorem 1. *Let M be a matroid on a set E with m elements and let w be a vector in the integer cone generated by the incidence vectors of the bases of M. Then w can be written as a nonnegative integer combination of at most $m + r(M) - 1$ incidence vectors of bases of M, where $r(M)$ is the rank of M.*

2. Definitions and preliminaries

Let M be a matroid on a set E. We denote the rank of a subset S of E by $r(S)$, or $r_M(S)$ in case of ambiguity. We sometimes write $r(M)$ for $r(E)$. If X and Y are disjoint subsets of E then $M/X\backslash Y$ denotes the matroid obtained from M by contracting X and deleting Y. By $M(X)$ we denote the restriction of M to X or the deletion of $E\backslash X$ from M.

Let w be a vector in $\mathbb{R}^E$. For each subset S of E we denote $\sum_{e \in S} w_e$ by $w(S)$. We denote by cone(M) and int.cone(M) the cone and the integer cone generated by the incidence vector of bases of M, respectively. It follows from the Edmonds' description of the matroid polytope by linear inequalities [4,5] that w belongs to cone(M) if and only if

$$w_e \geqslant 0 \quad \text{for each } e \text{ in } E, \tag{1}$$

$$w(S) \leqslant (w(E)/r(M))r(S) \quad \text{for each closed subset } S \text{ of } E.$$

We call a closed subset S of E w-tight if $w(S) = (w(E)/r(M))r(S)$. For a subset S of E, χ_S denotes the incidence vector of S. For each base B of M and w in cone(M) let $\mu_{w,B} := \max\{\mu \in \mathbb{R}_+ \colon\ w - \mu\chi_B \in \text{cone}(M)\}$.

3. Proof of the theorem

We first establish a key lemma from which the theorem follows easily.

Lemma 2. *Let M be a matroid on a set E and let w in* int.cone(M) *with $w_e > 0$ for each e in E and such that there is no w-tight proper subset of E. Suppose B_1 is a base of M such that μ_{w,B_1} is not an integer and let $v := w - \lfloor \mu_{w,B_1} \rfloor \chi_{B_1}$. Then there exists a base B_2* of *M and a proper subset X of E such that $\mu_{v,B_2} = 1$ and X is$(v - \chi_{B_2})$-tight.*

Proof. We may assume that $\mu_{w,B_1} < 1$ and, therefore, that $w = v$. Let B be a base of M such that $w - \chi_B$ belongs to int.cone(M), let $\{e_1, \ldots, e_l\} := B_1\backslash B$, let $\{f_1, \ldots, f_l\} := B\backslash B_1$, and let $k := w(E)/r(E) - 1$. We may assume that $T_i := B_1\backslash\{e_1, \ldots, e_i\} \cup \{f_1, \ldots, f_i\}$ is a base of M for $i = 0, 1, \ldots, l$. Since μ_{w,B_1} is not an integer, $\mu_{w,B_1} < 1$ and $\mu_{w,B} \geqslant 1$, there exists an index j in $\{1, \dots, l\}$ such that $\mu_{w,T_{j-1}} < 1$ and $\mu_{w,T_j} \geqslant 1$.

By hypothesis, $w_e > 0$ for each e in E and there is no w-tight proper subset of E. Thus, $\mu_{w,T_{j-1}} > 0$ and, by description (1) of cone(M), there exists a $(w - \mu_{w,T_{j-1}}\chi_{T_{j-1}})$-tight proper subset $X$of E. Hence,

$$w(X) - \chi_{T_{j-1}}(X) \geqslant kr(X) + 1. \tag{2}$$

On the other hand, as $\mu_{w,T_j} \geqslant 1$, we have that

$$w(X) - \chi_{T_j}(X) \leqslant kr(X). \tag{3}$$

Combining (2) and (3) we obtain that

$$\begin{aligned} kr(X) &\geqslant w(X) - \chi_{T_j}(X) \\ &= w(X) - \chi_{T_{j-1}}(X) + \chi_{\{e_j\}}(X) - \chi_{\{f_j\}}(X) \\ &\geqslant kr(X) + 1 + \chi_{\{e_j\}}(X) - \chi_{\{f_j\}}(X). \end{aligned}$$

So, we conclude that $e_j \notin X, f_j \in X$ and $w(X) - \chi_{T_j} = kr(X)$. One can also verify that $w - \chi_{T_j}$ belongs to int.cone(M). Hence, $B_2 := T_j$ is a base of M such that $\mu_{w,B_2} = 1$ and X is a $(w - \chi_{B_2})$-tight proper subset of E. $\square$.

We are now ready to prove Theorem 1.

Proof of Theorem 1. The proof is by induction on $m + r(M)$. If $r(M) = 1$, then we can trivially write w as a nonnegative integer combination of m incidence vectors of bases of M. So, we assume that $r(M) > 1$.

Let B_1 be a base of M and let $v := w - \lfloor \mu_{w,B_1} \rfloor \chi_{B_1}$. We consider two cases.

Case 1. μ_{w,B_1} is an integer.

By the description (1) of cone(M), there exists some e in E such that $v_e = 0$ or there exists a proper subset X of E which is v-tight.

Suppose that $E_0 := \{e \in E : v_e = 0\}$ is nonempty. Let $E' := E \backslash E_0$ and let v' be the restriction of v to E'. By induction hypothesis, we can write v' as a nonnegative integer combination of at most $m - |E_0| + r(E') - 1 \leqslant m + r(M) - 2$ incidence vectors of bases of $M(E')$. Hence, we can write w as a nonnegative combination of at most $m + r(M) - 1$ incidence vectors of bases of M.

So, we may assume that $E_0 = \emptyset$. Let X be a maximal proper v-tight set of E and let $Y := E \backslash X$. Let $M(X_1), \dots, M(X_p)$ be the connected components of $M(X)$, let v_Y be the restriction of v to Y, let v_{X_i} be the restriction of v to X_i $(i = 1, \dots, p)$, and let $k := w(E)/r(M) - \lfloor \mu_{w,B_1} \rfloor$. One can verify that:

(a) v_Y is the sum of k incidence vectors of bases of M/X;
(b) v_{X_i} is the sum of k incidence vectors of bases of $M(X_i)$ $(i = 1, \dots, p)$; and
(c) $r(M/X) < r(M)$ and $r(M(X_i)) < r(M)$ $(i = 1, \dots, p)$.

Thus, by induction, we can write v_Y as a nonnegative integer combination of at most $|Y| + r(M/X) - 1$ incidence vectors of bases of M/X and write v_{X_i} as a nonnegative integer combination of at most $|X_i| + r(M(X_i)) - 1$ incidence vectors of bases of $M(X_i)$ $(i = 1, \dots, p)$. These combinations can be glued together to form a nonnegative integer combination of v of at most

$$|Y| + \sum_{i=1}^{p} |X_i| + r(M/X) + \sum_{i=1}^{p} r(M(X_i)) - (p+1) - p = m + r(M) - 2p - 1$$

incidence vector of bases of M. This glueing can be done as follows. Let $B_{Y,1}, \dots, B_{Y,k}$ be bases of M/X such that:

- $v_Y = \chi_{B_{Y,1}} + \cdots + \chi_{B_{Y,k}}$;
- there exists indexes $0 = l_{Y,0} < l_{Y,1} < \cdots < l_{Y,k_Y} = k$ with $k_Y < |Y| + r(M/X) - 1$ such that $B_{Y,l_{Y,j}+1} = B_{Y,l_{Y,j}+2} = \cdots = B_{Y,l_{Y,j+1}}$ for $j = 0, \dots, k_Y - 1$.

For $i = 1, \dots, p$ let $B_{X_i,1}, \dots, B_{X_i,k}$ be bases of $M(X_i)$ such that:

- $v_{X_i} = \chi_{B_{X_i,1}} + \cdots + \chi_{B_{X_i,k}}$;
- there exists indexes $0 = l_{X_i,0} < l_{X_i,1} < \cdots < l_{X_i,k_{X_i}} = k$ with $k_{X_i} < |X_i| + r(M(X_i)) - 1$ such that $B_{X_i,l_{X_i,j}+1} = B_{X_i,l_{X_i,j}+2} = \cdots = B_{X_i,l_{X_i,j+1}}$ for $j = 0, \dots, k_{X_i} - 1$.

Then, by (a) and (b) above, $B_i := B_{Y,i} \cup B_{X_1,i} \cup \cdots \cup B_{X_p,i}$ is a base of M $(i = 1, \dots, k)$ and the sum $\chi_{B_1} + \cdots + \chi_{B_k}$ shows that v can be written as a nonnegative integer combination of at most

$$k_Y + \sum_{i=1}^{p} k_{X_i} - p \leqslant |Y| + \sum_{i=1}^{p} |X_i| + r(M/X) + \sum_{i=1}^{p} r(M(X_i)) - (p+1) - p$$

incidence vectors of bases of M.

Therefore, we can write w as a nonnegative integer combination of at most $m + r(M) - 2p \leqslant m + r(M) - 2$ incidence vectors of bases of M.

Case 2. μ_{w,B_1} is not an integer.

From Lemma 2, it follows that there exists a base B_2 of M and a proper subset X of E such that $\mu_{v,B_2} = 1$ and X is $(v - \chi_{B_2})$-tight. Now, proceeding similarly as in the Case 1, simply replacing w by v and B_1 by B_2, one can write v as a nonnegative integer combination of at most

$$m + r(M) - |E_0| - 2p \leqslant m + r(M) - 2,$$

incidence vectors of bases of M, where $E_0 = \{e \in E \colon v_e - \chi_{B_2}(e) = 0\}$ and p is the number of connected components of $M(X)$.

Thus, we can write w as a nonnegative integer combination of at most $m + r(M) - 1$ incidence vectors of bases of M. This completes the proof of theorem. $\square$.

4. Final remarks

By making use of the Discrete Separation Theorem [6,7] and of an efficient algorithm to minimize a submodular set function [8,10] one can show that the proof of Theorem 1 yields an efficient algorithm. We also mention the NP-completeness of the following related problem: *given*: a graph G with vertex set $\{u, v, w\}$ and edge set $\{a, b, e_1, \ldots, e_n\}$ such that a and b connect u and v and e_i connects v and w $(i = 1, \ldots, n)$, a vector w in the integer cone of the spanning trees of G, and an integer t; *question*: can w be written as a nonnegative integer combination of at most t incidence vectors of spanning trees of G? The PARTITION problem can be reduced to this problem.

A theorem (and lemma) similar to the one presented in this paper has been proved by Pevsner for rooted arborescences (cf. Sebő [11]). It could be interesting to find a common generalization of the two results. The Integer Carathéodory problem remains open for both of these cases.

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